International Journal of Theoretical Physics, Vol. 27, No. 6, 1988

Symmetry Vector Fields and Similarity Solutions of a Nonlinear Field Equation Describing the Relaxation to a Maxwell Distribution

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Received October 27, 1987

In the study of the formulation of Maxwellian tails the nonlinear partial differential equation $\partial^2 u/\partial x\, \partial\tau + \partial u/\partial x + u^2 = 0$ arises. We determine the Lie point symmetry vector fields and calculate the similarity ansätze. Then we discuss the resulting ordinary differential equations. Finally, the existence of Lie Bäcklund vector fields is studied and a Painlevé analysis is performed.

In the investigation of the formation of Maxwellian tails the following nonlinear partial differential equation arises:

$$\frac{\partial^2 u}{\partial x\, \partial \tau} + \frac{\partial u}{\partial x} + u^2 = 0 \tag{1}$$

The derivation of this equation is as follows: The state of the gas at a dimensionless time τ is described by a distribution function $nf(v, \tau)$, where n is the constant number density, $\mathbf{v}$ is a velocity variable, and $v = |\mathbf{v}|$. The Boltzmann equation in a simplified form takes the form

$$\frac{\partial f(v, \tau)}{\partial \tau} = -f(v, \tau) + \frac{1}{4\pi} \int d^3 w \int_0^{\pi} d\chi \sin\chi \int_0^{2\pi} d\varepsilon\, f(v', \tau) f(w', \tau) \tag{2}$$

with

$$v'^2 = \tfrac{1}{2}(v^2 + w^2) - \tfrac{1}{2}(v^2 - w^2)\cos\chi + |\mathbf{v} \times \mathbf{w}| \sin\chi \cos\varepsilon \tag{3}$$

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0020-7748/88/0600-0717$06.00/0

and

$$w'^2 = \tfrac{1}{2}(v^2 + w^2) - \tfrac{1}{2}(v^2 - w^2)\cos\chi - |\mathbf{v}\times\mathbf{w}|\sin\chi\cos\varepsilon \tag{4}$$

Normalized moments $M_k(\tau)$ of f are defined by the equation

$$M_k(\tau) \equiv \frac{\sqrt{\pi}}{2(2\beta^2)^k\Gamma(k+3/2)}\int v^{2k}f(v,\tau)\,d^3v \tag{5}$$

with

$$M_0(\tau) \equiv 1, \qquad M_1(\tau) \equiv 1 \tag{6}$$

and

$$M_k(\infty) = 1 \tag{7}$$

where $k = 0, 1, \ldots$. Multiplication of (2) by v^{2k} and integration over $\mathbf{v}$ space leads to the infinite sequence of moment equations

$$\frac{dM_k}{d\tau} + M_k = \frac{1}{k+1}\sum_{m=0}^{k} M_m M_{k-m} \tag{8}$$

Introducing the generating function $G(\xi, \tau)$ for the normalized moments,

$$G(\xi, \tau) \equiv \sum_{k=0}^{\infty} \xi^k M_k(\tau) \tag{9}$$

yields

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial G}{\partial\tau} + \xi G\right) = G^2 \tag{10}$$

The transformation

$$x = \frac{1-\xi}{\xi}, \qquad u(x,\tau) = \xi G(\xi,\tau) \tag{11}$$

leads to (1). Equation (1) has been derived by Krook and Wu (1976). They also gave one similarity ansatz and the corresponding similarity solution. We determine the Lie point symmetry vector fields for (1). Then we calculate different similarity ansätze. The resulting ordinary differential equations are discussed. Finally, the existence of Lie Bäcklund vector fields is investigated and a Painlevé test is performed.

First we determine the Lie point symmetry vector fields. For describing Lie point symmetry vector fields the jet bundle technique is a suitable approach (Steeb, 1984; Steeb and Strampp, 1982). We consider the submanifold

$$F \equiv u_{x\tau} + u_x + u^2 = 0 \tag{12}$$

and all its differential consequences with respect to the space coordinate. This means

$$F_x \equiv u_{xx\tau} + u_{xx} + 2uu_x = 0 \tag{13}$$

and so on. Let

$$U = a(x, \tau, u)\frac{\partial}{\partial x} + b(x, \tau, u)\frac{\partial}{\partial \tau} + c(x, \tau, u)\frac{\partial}{\partial u}$$

be a Lie point symmetry vector field. Then the corresponding vertical vector field is given by

$$V = (-u_x a - u_\tau b + c)\frac{\partial}{\partial u} \tag{14}$$

The invariance requirement (this means V is a symmetry vector field of $F = 0$) is expressed as

$$L_{\bar{V}} F \triangleq 0 \tag{15}$$

where $L_{\bar{V}}(\cdot)$ denotes the Lie derivative and $\triangleq$ stands for the restriction to solution of (1). The field $\bar{V}$ is the extended vector field of V. Due to the structure of (1), we have only to include the terms of the form $(\cdots)\,\partial/\partial u_x$ and $(\cdots)\partial/\partial u_{x\tau}$ in the extended vector field $\bar{V}$. Separating out the terms with the coefficients $u_{\tau\tau}$, u_{xx}, $u_\tau u_{xx}$, $u_x u_{\tau\tau}$, $u_\tau u_x^2$, $u_x u_\tau^2$, u_x^2, u_τ^2, $u_x u_\tau$, and 1, we find from the invariance requirement (15)

$$\begin{gathered}
\frac{\partial b}{\partial x} = \frac{\partial a}{\partial \tau} = \frac{\partial a}{\partial u} = \frac{\partial b}{\partial u} = 0 \\
\frac{\partial^2 a}{\partial u^2} = \frac{\partial^2 b}{\partial u^2} = 0 \\
\frac{\partial a}{\partial u} + \frac{\partial^2 a}{\partial u\,\partial \tau} = 0 \\
\frac{\partial^2 b}{\partial x\,\partial u} = 0 \\
\frac{\partial^2 a}{\partial x\,\partial u} + \frac{\partial^2 b}{\partial \tau\,\partial u} - \frac{\partial b}{\partial u} - \frac{\partial^2 c}{\partial u^2} = 0 \\
\frac{\partial b}{\partial x} + \frac{\partial^2 b}{\partial x\,\partial \tau} - \frac{\partial^2 c}{\partial u\,\partial x} - 2u^2\frac{\partial b}{\partial u} = 0 \\
\frac{\partial^2 a}{\partial x\,\partial \tau} - 2u^2\frac{\partial a}{\partial u} - \frac{\partial b}{\partial \tau} - \frac{\partial^2 c}{\partial \tau\,\partial u} = 0 \\
\frac{\partial c}{\partial x} + \frac{\partial^2 c}{\partial x\,\partial \tau} - u^2\frac{\partial c}{\partial u} + u^2\frac{\partial a}{\partial x} + u^2\frac{\partial b}{\partial \tau} + 2uc = 0
\end{gathered} \tag{16}$$

Solving this set of partial differential equations, we find four symmetry vector fields, namely

$$X=\frac{\partial}{\partial x}, \qquad T=\frac{\partial}{\partial \tau}$$

$$S=-x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u} \tag{17}$$

$$G=e^{\tau}\frac{\partial}{\partial \tau}-e^{\tau}u\frac{\partial}{\partial u}$$

The symmetry vector fields form a Lie algebra, as they must. We find

$$[X, T]=0, \qquad [X, S]=-X$$

$$[X, G]=0, \qquad [T, S]=0 \tag{18}$$

$$[T, G]=G, \qquad [S, G]=0$$

Let us now introduce similarity ansätze with the help of the symmetry vector fields given by (17). The simplest similarity ansatz is given by the symmetry vector fields $\partial/\partial x$ and $\partial/\partial \tau$. We then find the ansatz $u(x, \tau)=f(s)$, where $s=c_1 x+c_2 \tau$ is the similarity variable. Inserting this ansatz into (1) yields

$$c_1 c_2 \frac{d^2 f}{ds^2}+c_1\frac{df}{ds}+f^2=0 \tag{19}$$

where $c_1 c_2 \neq 0$. The solution of this equation cannot be given explicitly. Thus, the equation is not integrable. Let us now consider the symmetry vector field

$$V=c_1\frac{\partial}{\partial x}+c_2\frac{\partial}{\partial \tau}+c_3\left(-x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}\right) \tag{20}$$

where $c_1, c_2, c_3 \in \mathscr{R}$. The autonomous system of first-order ordinary differential equations

$$\frac{d\tau}{d\varepsilon}=c_2, \qquad \frac{dx}{d\varepsilon}=c_1-c_3 x, \qquad \frac{du}{d\varepsilon}=c_3 u \tag{21}$$

leads to the transformation group

$$\tau(\varepsilon)=c_2\varepsilon+\tau_0$$

$$x(\varepsilon)=\frac{c_1}{c_3}-\frac{c_1-c_3\ x_0}{c_3}\ e^{-c_3\varepsilon} \tag{22}$$

$$u(\varepsilon)=u_0\, e^{c_3\varepsilon}$$

where $c_3 \neq 0$. Now let $\tau_0 = s/c$ and $x_0 = 1$ with $c \neq 0$. Then we obtain the similarity variable

$$s = c\tau + \frac{c_2 c}{c_3} \ln \frac{c_3 x - c_1}{c_3 - c_1} \tag{23}$$

and the similarity ansatz

$$u(x, \tau) = f(s) \frac{c_1 - c_3}{c_1 - c_3 x} \tag{24}$$

We now study the simplified case $c_1 = 0$ and $c_2 c / c_3 = 1$. Then we obtain the ordinary differential equation

$$cf'' + (1-c)f' - (1-f)f = 0 \tag{25}$$

where $f' \equiv df(s)/ds$. Now let $F(f(s)) := df(s)/ds$. Consequently

$$cF\frac{dF}{df} + (1-c)F - (1-f)f = 0 \tag{26}$$

Boundary conditions corresponding to (6) and (7) determine that $c = \frac{1}{6}$ and therefore the explicit solution (Krook and Wu, 1976) of (26) is

$$F(f(s)) = 2(1-f)[1-(1-f)^{1/2}] \tag{27}$$

Equation (1) can be simplified using the transformation

$$\begin{aligned} \tilde{X}(x, \tau, u(x, \tau)) &= x \\ \tilde{T}(x, \tau, u(x, \tau)) &= -e^{-\tau} \\ \tilde{U}(\tilde{X}(x, \tau, u(x, \tau))\tilde{T}(x, \tau, u(x, \tau))) &= e^{\tau} u(x, \tau) \end{aligned} \tag{28}$$

We thus obtain the equation

$$\frac{\partial^2 \tilde{U}}{\partial \tilde{X}\, \partial \tilde{T}} + \tilde{U}^2 = 0 \tag{29}$$

where $-1 \leq \tilde{T} < 0$, since $0 \leq \tau < \infty$. Under the transformation (28) the Lie symmetry vector fields transform as follows:

$$\begin{aligned} \frac{\partial}{\partial x} &\to \frac{\partial}{\partial \tilde{X}} \\ \frac{\partial}{\partial \tau} &\to -\tilde{T}\frac{\partial}{\partial \tilde{T}} + \tilde{U}\frac{\partial}{\partial \tilde{U}} \\ +x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} &\to -\tilde{X}\frac{\partial}{\partial \tilde{X}} + \tilde{U}\frac{\partial}{\partial \tilde{U}} \\ e^{\tau}\left(\frac{\partial}{\partial \tau} - u\frac{\partial}{\partial u}\right) &\to \frac{\partial}{\partial \tilde{T}} \end{aligned} \tag{30}$$

Let us mention that the equation $\partial^2\tilde{U}/\partial\tilde{X}\,\partial\tilde{T}+\tilde{U}^n=0(n=2,3,\ldots)$ admits the following symmetry vector fields:

$$\left\{\frac{\partial}{\partial\tilde{X}};\frac{\partial}{\partial\tilde{T}};-\tilde{T}\frac{\partial}{\partial\tilde{T}}+\frac{\tilde{U}}{n-1}\frac{\partial}{\partial\tilde{U}};-\tilde{X}\frac{\partial}{\partial\tilde{X}}+\frac{\tilde{U}}{n-1}\frac{\partial}{\partial\tilde{U}}\right\}$$

The simplest similarity ansatz for (29) is

$$\tilde{U}(\tilde{X},\tilde{T})=f(c_1\tilde{X}+c_2\tilde{T}) \tag{31}$$

which is generated by the vector fields $\partial/\partial\tilde{X}$ and $\partial/\partial\tilde{T}$. Then we find

$$c_1c_2\frac{d^2f}{ds^2}+f^2=0 \tag{32}$$

where $s=c_1\tilde{X}+c_2\tilde{T}(c_1,c_2\neq 0)$. Introducing the scaling $\bar{s}=(6c_1c_2)^{-1/2}s$ and $\bar{f}=-f$ yields

$$\frac{d^2\bar{f}}{d\bar{s}^2}-6\bar{f}^2=0 \tag{33}$$

The general solution can be written in the form

$$\bar{f}(\bar{s})=c^2\left[\frac{-k^2}{1+k^2}+\frac{1}{\mathrm{sn}^2\{c(\bar{s}-\bar{s}_1),k\}}\right] \tag{34}$$

where c and $\bar{s}_1$ are arbitrary constants, sn is an elliptic function, and k^2 is a root of the equation $1-k^2+k^4=0$. Then the solution to (1) is given by

$$\bar{f}(\bar{s})=-c^2\left[\frac{-k^2}{1+k^2}+\frac{1}{\mathrm{sn}^2\{c(6c_1c_2)^{-1/2}(c_1x-c_2e^{-\tau}-c_3),k\}}\right] \tag{35}$$

where c, c_1, c_2, and c_3 are arbitrary constants.

The existence of the Lie Bäcklund vector fields for equations of the type

$$\frac{\partial^2\tilde{U}}{\partial\tilde{X}\,\partial\tilde{T}}=F(\tilde{U}) \tag{36}$$

has been investigated by Steeb (1984). It turns out that $\partial^2\tilde{U}/\partial\tilde{X}\,\partial\tilde{T}+\tilde{U}^n=0$ $(n=2,3,\ldots)$ does not admit Lie Bäcklund vector fields. We have also performed the Painlevé test, due to Weiss *et al.* (1983). Inserting the ansatz $u\propto u_0\phi^n$ into (1) yields $n=-2$ and $u_0=-6\phi_\tau\phi_x$. The resonances are given by -1 and 6. Using computer algebra, we find that (1) does not pass the Painlevé test. With the reduced ansatz $\phi(x,\tau)=x-f(\tau)$ we find at the resonance $r=6$ that the condition $F(f,df/d\tau,\ldots)=0$ is not satisfied identically. The field equation $\partial^2\tilde{U}/\partial\tilde{X}\,\partial\tilde{T}+\tilde{U}^n=0$ also does not pass the Painlevé test for all $n=2,3,\ldots$. The only differential equation that passes the Painlevé test is the ordinary differential equation (32). This equation also has the Painlevé property (Davis, 1962).

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